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A SECRETARY PROBLEM WITH UNCERTAIN EMPLOYMENT

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Abstract

A 'Secretary Problem' with no recall but which allows the applicant to refuse an offer of employment with a fixed probability $1-p$, $(0 < p < 1)$, is considered. The optimal stopping rule and the maximum probability of employing the best applicant are derived.

SEQUENTIAL DECISION THEORY; OPTIMAL STOPPING RULES; SECRETARY PROBLEMS

1. Introduction and summary

In this note we consider a variation of the problem treated under such names as 'Googol', the 'Secretary Problem' and the 'Marriage Problem', in, for example, Fox and Marnie (1960), Lindley (1961) and Gilbert and Mosteller (1966). A known number, N, of applicants for a single position are presented in random order to an employer who observes the rank of the present applicant relative to those preceding her. At each stage the employer must decide whether to employ the present applicant (she accepts an offer with certainty) or to continue to interview further applicants. (There is no recall of applicants already passed over.) The optimal stopping rule which maximises the probability of employing the best applicant is well known.

The problem we are to consider allows the applicant the right to refuse an offer of employment. We assume she accepts an offer of employment with a known probability p, $(0 < p < 1)$, independent of her rank and the disposition of the other applicants. For ease of formulation as a stopping rule problem, we assume that the employer will ascertain the availability of the applicant at each stage. It is noted in Remark 1 that in fact the employer need only ascertain the availability of an applicant when he would employ her if she were available. He can therefore ascertain her availability by offering her the position and stopping if she accepts. In Section 2 it is shown that a stopping rule which maximises the probability of employing the best girl is :

pass over the first $r^* - 1$ applicants and thereafter stop with the first available applicant who is better than all those preceding her, where r^* is the smallest integer r in $\{1, 2, \cdots, N-1\}$ such that

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(1) $$\prod_{k=r}^{N-1}\left(1+\frac{1-p}{k}\right)\leqq\frac{1}{p}.$$

The maximum probability of employing the best applicant is given by

(2) $$v_0=\frac{p}{(1-p)N}\left[(r^*-p)\prod_{k=r^*}^{N-1}\left(1+\frac{1-p}{k}\right)-r^*+1\right].$$

It is also shown that,

(3) $$\lim_{N\to\infty}\frac{r^*}{N}=\lim_{N\to\infty}v_0=p^{\,1/(1-p)}.$$

Table 1 gives values of r^* and v_0 (truncated at 5 decimal places) for various values of p and N, the values for $p = 1$ being taken from Table 2 in Gilbert and Mosteller (1966).

TABLE 1

N	$p = 0.5$		$p = 0.9$		$p = 1$	
	r^*	v_0	r^*	v_0	r^*	v_0
2	1	.37500	1	.49500	1	.50000
3	1	.31250	2	.46500	2	.50000
4	2	.29687	2	.43537	2	.45833
5	2	.29218	3	.40365	3	.43333
10	3	.26985	4	.37936	4	.39869
25	7	.25770	10	.36024	10	.38091
50	13	.25379	18	.35460	19	.37427
100	26	.25187	36	.35161	38	.37104
1000	251	.25018	349	.34897	369	.36819
∞	$.25N$	.25000	$.34867N$	.34867	$.36787N$	.36787

2. Derivation of results

Let X_r be the rank of applicant r relative to those preceding her. Also let Y_r be a random variable taking a value of either 1 or 0 according to whether applicant r is available or not. It is assumed that the availabilities of the applicants are independent of one another and of the ranks of the applicants. Thus $X_1, Y_1, X_2, Y_2, \cdots, X_N, Y_N$ are independent random variables with distributions given by

$$P(X_r = k) = 1/r, \qquad k = 1,2,\cdots,r,$$

and

$$P(Y_r = i) = p^i(1-p)^{1-i}, \qquad i = 0, 1.$$

The problem is to find a stopping rule t^* among all stopping rules on (X_1, Y_1), $(X_2, Y_2), \cdots, (X_N, Y_N)$ which maximises the probability of employing the best applicant. Using a notation similar to that used by Lindley (1961), we denote by $\bar{U}_r(k, i)$ the utility of stopping at stage r when $X_r = k$ and $Y_r = i$. (By stage r we mean immediately after observing X_r, Y_r.) Thus

$$\bar{U}_r(k,i) = \begin{cases} r/N, & k = 1,\ i = 1, \\ 0, & k, i \text{ otherwise}, \end{cases} \tag{4}$$

is the probability applicant r is best given $X_r = k$ and $Y_r = i$. The backward induction equations for generating t^* are

$$V_N(k,i) = \bar{U}_N(k,i), \qquad k = 1, 2, \cdots, N,\ i = 0, 1, \tag{5}$$

$$V_r(k,i) = \max\{\bar{U}_r(k,i), E[V_{r+1}(X_{r+1}, Y_{r+1})]\}, \tag{6}$$
$$r = 1, 2, \cdots, N-1,\ k = 1, 2, \cdots, r,\ i = 0, 1.$$

It should be noted that technically the expectation in (6) is conditional upon the previous observations but since the observations are independent this conditioning is unnecessary. If we set

$$v_r = E[V_{r+1}(X_{r+1}, Y_{r+1})], \qquad r = 0, 1, \cdots, N-1, \tag{7}$$

then it is clear that t^* is given by: stop at stage r if and only if $X_r = 1$, $Y_r = 1$ and $v_r \leqq r/N$. Also v_0 is the maximum probability of employing the best applicant. (5) and (6) become

$$v_{N-1} = p/N, \tag{8}$$

$$v_{r-1} = v_r + (r/N - v_r)^+ p/r, \qquad r = 1, 2, \cdots, N-1. \tag{9}$$

Thus v_r is decreasing in r and r/N is increasing in r and hence there exists an $r^* \in \{1, 2, \cdots, N-1\}$ such that $v_r \leqq r/N$ if and only if $r \geqq r^*$.

The solution of (9) subject to boundary condition (8) is

$$v_r = \frac{pr}{(1-p)N}\left(\prod_{k=r}^{N-1}\left(1 + \frac{1-p}{k}\right) - 1\right), \quad r = r^* - 1, r^*, \cdots, N-1, \tag{10}$$

and

$$v_r = v_{r^*-1}, \qquad r = 0, 1, \cdots, r^* - 1.$$

Hence r^* is as given in Section 1 and $v_0 = v_{r^*-1}$ is as given by (2) after rearranging (10) to avoid the awkwardness when r^* happens to be 1.

Simple bounds for r^* can be obtained from the inequalities

$$\left(\frac{k+1}{k}\right)^{1-p} < 1 + \frac{1-p}{k} < \left(\frac{k+1-p}{k-p}\right)^{1-p}.$$

The definition of r^* therefore requires that

$$\left(\frac{N}{r^*}\right)^{1-p} < \frac{1}{p} < \left(\frac{N-p}{r^*-1-p}\right)^{1-p},$$

and hence

$$Np^{1/(1-p)} < r^* < Np^{1/(1-p)} + 1 + p(1 - p^{1/(1-p)}).$$

Thus

$$\lim_{N\to\infty} r^*/N = p^{\,1/(1-p)},$$

and using (10) we have

$$\lim_{N\to\infty} v_0 = \lim_{N\to\infty} v_{r^*-1} = \lim_{N\to\infty} v_{r^*} = p^{1/(1-p)}.$$

3. Remarks

1. Since the Y_r are independent of each other and of the X_r, it is only necessary to observe Y_r if both $r \geqq r^*$ and $X_r = 1$. The optimal stopping rule is then: offer the position to each applicant, from stage r^* onwards, who is relatively best; if she accepts, stop, otherwise go on.

2. It would be reasonable to allow the probability of acceptance, p, to be a decreasing function of the absolute rank of the applicant. However this will result in (X_r, Y_r) being dependent upon previous observations with the consequent complication of the form of the optimal stopping rule.

3. Uncertainty of employment could be extended to the minimisation of expected rank problem considered by Lindley (1961) and Chow *et al.* (1964) and also to the classes of problems considered by Gusein-Zade (1966) and Mucci (1973a) and (1973b).

4. Yang (1974) investigates a class of secretary problems which permit the offering of employment to applicants already passed over, but with decreasing probability of acceptance. Nevertheless, he considers only situations where the present applicant will accept an offer with certainty.

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